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CAVALIERI'S THEOREM IN HIS OWN WORDS.

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Introduction. Cavalieri's Theorem has recently come into some use in the teaching of elementary geometry as a means of unifying the ideas that lie at the basis of the mensuration of solids. The substance of the theorem is as follows:

If two solids have equivalent bases, and if sections parallel to the bases and equally distant from them in the two solids are also equivalent, then the solids are equivalent.

It is natural to think of the two solids referred to as being generated by two surfaces constantly equivalent to each other and constantly parallel to the respective bases, and we have an instinctive willingness to accept the statement that the two solids, so generated, are equivalent.

It is possible, however, to prove the theorem with as much rigor as is looked for in any demonstration involving the theory of limits; and it would be desirable, if it were possible, to prove this theorem at the beginning of the mensuration work of solid geometry. We then should be able to do away with the somewhat vexatious theorem about the equivalence of two parallelepipeds having equivalent bases and equal altitudes; and also to do away with the theorem about the equivalence of two pyramids having equivalent bases and equal altitudes. Moreover, the proof of the volume of a cylinder, or even of a cone, could be made, by means of Cavalieri's theorem, to depend directly upon the theorems in plane geometry about the area of a circle.

The more one examines the objects attainable by the use of this theorem as a basic proposition of mensuration, the more attractive it looks, especially in the matter of diminishing recourse to the theory of limits. On that account, the proof given by Cavalieri himself nearly three hundred years ago will doubtless be of interest to students and teachers of solid geometry. The diagram is a substantially exact copy of Cavalieri's own drawing, but the lettering has been

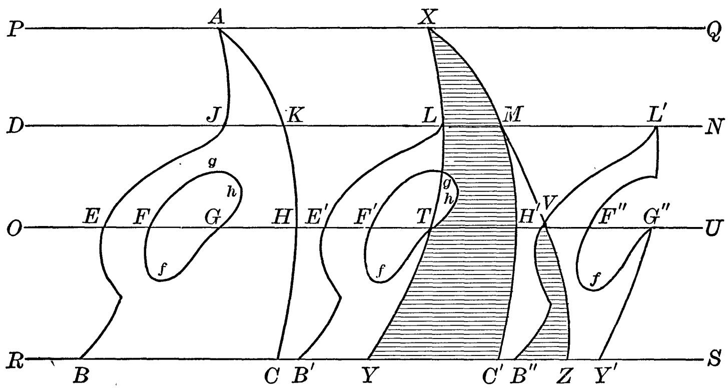
changed to take advantage of the modern habit of lettering a diagram so as to give some clue to the relations of the figure. The translation is intended to give, as faithfully as possible, the verbal meaning of the Latin, while not, of course, following the prolix idiom of the time in all its ramifications.

The book from which the following is taken is *Geometria Indivisibilibus Continuorum Nova quadam ratione promota. Authore P. Bonaventura Cavalerio.* . . . Bononiæ MDCLIII.

The Theorem.[1] *If between the same parallels any two plane figures are constructed, and if in them, any straight lines being drawn equidistant from the parallels, the included portions of any one of these lines are equal, the plane figures are also equal to one another; and, if between the same parallel planes any solid figures are constructed, and if in them, any planes being drawn equidistant from the parallel planes, the included plane figures out of any one of the planes so drawn are equal, the solid figures are likewise equal to one another.*

The figures so compared let us call analogues,[2] *the solid as well as the plane.* . . .

The Proof. Let any two plane figures ABC and XYZ be constructed be-



tween the same parallels PQ, RS; and let DN, OU, be drawn parallel to the aforesaid PQ, RS; and let the portions, *e. g.*, of DN, included in the figures, namely JK, LM, be equal to each other; and again, in the line OU, let the portions EF, GH, taken together (for the figure ABC, *e. g.*, may be hollow within, according to the contour of FgG), be likewise equal to TV; and let this happen in all the other lines equidistant from PQ. I say that the figures, ABC, XYZ, are equal to each other.

[1] FIGURÆ PLANÆ QUÆCUMQUE IN EISDEM PARALLELIS CONSTITUTÆ IN QUIBUS, DUCTIS QUIBUSCUMQUE EISDEM PARALLELIS ÆQUIDISTANTIBUS RECTIS LINEIS, CONCEPTÆ CUIUSCUMQUE RECTÆ LINEÆ PORTIONES SUNT ÆQUALES, ETIAM INTER SE ÆQUALES ERUNT; ET FIGURÆ SOLIDÆ QUÆCUMQUE IN EISDEM PLANIS PARALLELIS CONSTITUTÆ, IN QUIBUS, DUCTIS QUIBUSCUMQUÆ PLANIS EISDEM PLANIS PARALLELIS ÆQUIDISTANTIBUS, CONCEPTÆ CUIUSCUMQUE SIC DUCTI PLANI IN IPSIS SOLIDIS FIGURÆ PLANÆ SUNT ÆQUALES, PARITER INTER SE ÆQUALES ERUNT. . . . Op. cit., p. 484.

[2] Æqualiter analogæ.

Let either, then, of the two figures ABC, XYZ be taken, for example ABC itself, with the portions of the parallels PQ, RS coterminous with it, namely the portions PA, RB, and let it be superposed upon the other figure XYZ, but so that the lines PA, RB may fall upon AQ, CS; then either the whole figure ABC coincides with the whole figure XYZ (and thus, since they coincide with each other they are equal), or not; yet let there be some part which will coincide with some part, as $XMC'YThL$, part of the figure ABC, with $XMC'YThL$, part of the figure XYZ.

It is manifest, moreover, if the superposition of the figures is effected in such a way that portions of the parallels PQ, RS coterminous with our two figures are mutually superposed, that whatever straight lines (included in the figures) are in line, remain in line; as, for example, since EF, GH are in line with TV, when the aforesaid superposition is made they will remain in line (namely $E'F'TH'$ in line with TV), for the distance of those lines EF, GH from PQ is equal to the distance of TV from PQ; whence, no matter how many times PA is placed over AQ, at any place, EF, GH will always remain in line with TV, which is clearly apparent not only for this but for all other lines parallel to PQ in either figure.

In the case where part of one figure (as ABC) coincides of necessity with part of the figure XYZ, and not with the whole, granting that the superposition be made by such a rule as has been told, the demonstration will be as follows. For since when any parallels are drawn to PQ, the portions of them, included in the figures, which were in line, will still remain in line after superposition, and moreover since they were by hypothesis equal before superposition, therefore, after superposition the portions included in the figures will likewise be equal—as, *e. g.*, $E'F'$, TH' taken together will be equal to TV—therefore, if $E'F'$, TH' do not coincide with the whole of TV, then, one part [of one] coinciding with some part [of the other], as TH' with TH' itself, $E'F'$ will be equal to $H'V$, $E'H'$ being in the residuum of the figure ABC which is superposed, and $H'V$ in the figure XYZ upon which the other is superposed. In the same way we shall show that to any line whatever parallel to PQ, and included in the residuum of the superposed figure ABC (which may be $LB'YTF'$) corresponds an equal straight line, in line [with the former], which will be in the residuum of the figure XYZ on which ABC is superposed; therefore, the superposition being made by this rule, when anything of the superposed figure is left over and does not fall upon the figure, it must be that something of the other figure must also be left over, and have nothing superposed upon it.

Since, moreover, to each of the straight lines parallel to PQ and included in the residuum (or residua, for there may be several residual figures) of the superposed figure ABC (or $XB'C'$) there corresponds another straight line, in line [with the first] and included in the residuum (or residua) of the figure XYZ, it is manifest that these residual figures, or their aggregates, are between the same parallels; so since the residual figure $LB'YTF'$ is between the parallels DN, RS, likewise the residual figure (or aggregate of residual figures) of the figure XYZ

(because it has the frusta Thg, $MC'Z$) will be between the same parallels DN, RS. For if it did not extend both ways to the parallels DN, RS, as for example if it extended up to DN, but not down to RS, only as far as OU, then to the straight lines included in the frustum $E'B'YfF'$, and parallel to PQ, there would not be found in the residuum of the figure XYZ (or in the aggregate of the residua) other corresponding lines as has been proved to be unavoidable. Therefore these residua, or their aggregates, are between the same parallels; and the portions of the lines parallel to PQ, RS, included therein, are equal, as we have shown above; therefore the residua are subject to the same condition as has been assumed for ABC, XYZ; that is, they are analogues.

So let the residua be now superposed, but so that the parallels KL, CY may fall upon the parallels LN, YS, and the part $VB''Z$ of the frustum $LB'YTF'$ may coincide with the part $VB''Z$ of the frustum $MC'Z$; then we shall show, as above, that as long as there is found a residuum of one, there will be found also a residuum of the other, and these residua, or aggregates of residua, will be found within the same parallels. Let $L'VZY'G''F''$ be a residuum belonging to the figure ABC; and let $MC'B''V$, Thg, be residua belonging to the figure XYZ, whose aggregate is between the same parallels as the residuum $L'VZY'G''F''$, that is, between DN, RS. If now we superpose these residua again, but so that the parallels between which they lie be always superposed respectively, and this is supposed to be done continually, until the whole figure ABC shall have been superposed, I say the whole of it must coincide with XYZ; otherwise if there were any residuum of the figure XYZ, upon which nothing is superposed, there would be also some residuum of the figure ABC which would not have been superposed, as we have shown above to be unavoidable; but it is granted that the whole of ABC is superposed upon XYZ, therefore they are so superposed upon each other that there are no residua of either, therefore they are so superposed that they coincide, therefore the figures ABC, XYZ are equal to each other.

Now in the same diagram let ABC, XYZ be any two solid figures constructed between the same parallel planes PQ, RS; and let DN, OU be any planes drawn equidistant from the planes previously spoken of; and let the figures that lie in the same plane and that are included in the solids be equal to each other always; as JK equal to LM, and EF, GH, taken together (for a solid figure, for example ABC, may be hollow in any way within, according to the surface $FfGg$), equal to TV. I say that these solid figures are equal to each other.

For if we superpose the solid ABC, with the portions PA, RC of the planes PQ, RS, coterminous with it, upon the solid XYZ, in such a way that the plane PA be on the plane PQ, and the plane RC on the plane RS, we shall show (as we did above about the portions of the lines parallel to PQ included in the plane figures ABC, XYZ) that the figures included in the solids and lying in the same plane will also after superposition remain in the same plane; and therefore thus far the figures included in the superposed solids are equal,—and parallel to PQ, RS.

Then unless the entire solid coincides with the other solid entire in the first

superposition, residual solids will remain, or solids composed of residua, in either solid, which will not be superposed upon each other. Because since for example the figures $E'F'$, TH' are equal to the figure TV, then when the common figure TH' is taken away, the remaining figure $E'F'$ will be equal to the remaining figure $H'V$; and this will happen in any plane whatever parallel to PQ and meeting the solids ABC, XYZ. Therefore whenever we have a residuum of one solid, we shall always have a residuum of the other also; and it will be evident, according to the method applied in the former part of this Proposition in the case of plane figures, that the residua of the solids, or the aggregates of residua, will always be between the same parallel planes (as the residua $LB'YTF'$, $MC'Z$, Thg are between the same parallels DN, RS) and will be analogues.

Now if these residua be superposed again, so that the plane DL will be placed on the plane LN, and RY on YS, and this is understood to be done continually, until ABC, which is being superposed, is entirely taken, the entire solid ABC will finally coincide with the entire solid XYZ. For when the entire solid ABC is superposed upon XYZ, unless they coincided there would be some residuum of one, as of the solid XYZ, therefore also some residuum of the solid $XB'C'$, or ABC, and this residuum would not be superposed; which is absurd, for it is already assumed that the entire solid ABC is superposed on XYZ. Therefore there will not be any residuum in these solids; therefore they will coincide; therefore the solid figures spoken of, ABC, XYZ, will be equal to each other, which was to be proved of them.

MATHEMATICAL FORMS OF CERTAIN ERODED MOUNTAIN SIDES.

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In some of the desert regions of the west, visited by occasional heavy rainstorms, the formation is such that erosion takes place in a way that makes it possible to calculate approximately the form of the curve of intersection of the eroded slope and the alluvial fan of the plain.

This problem has been suggested by Professor A. C. Lawson of the department of geology of the University of California. He has formulated the hypotheses used below and the forms of the curves here obtained have been closely approximated by the actual geological conditions observed by him.

In the figure $S_0OR_0T_0$ is the contour of valley, hillside and plateau at some initial time. The table land R_0T_0 may be taken as inclined at a small angle C to the horizontal direction T_0Y. The valley OS_0 is assumed to be inclined at an angle B. The eroded material OR_0RH comes off in layers and is deposited in the valley in the position S_0SHO. The slope of erosion as well as the floor of the valley maintains, in the observed regions, a fairly constant inclination. Both inclinations are assumed constant in these calculations. The line S_0S, at the low